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ON THE CONCEPT OF LEVEL FOR SUBGROUPS OF SL_2 OVER ARITHMETIC RINGS

BY

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ABSTRACT

We define the concept of level for arbitrary subgroups Γ of finite index in the special linear group $SL_2(A_S)$, where A_S is the ring of S-integers of a global field k provided that k is an algebraic number field, or card $(S) \geq 2$. It is shown that this concept agrees with the usual notion of 'Stufe' for congruence subgroups. In the case $SL_2(\mathcal{O})$, $\mathcal{O}$ the ring of integers of an imaginary quadratic number field, this criterion for deciding whether or not an arbitrary subgroup of finite index is a congruence subgroup is used to determine the minimum of the indices of non-congruence subgroups.

1. Introduction

In the theory of modular forms, the concept of level for an arbitrary subgroup Γ of finite index in $SL_2(\mathbb{Z})$ was defined by Wohlfahrt [21, 22] as the least common multiple of the widths of the cusps of the fundamental domain for the group. Previously the notion of level, called 'Stufe' by Klein, was only defined for congruence subgroups. It is a result of Fricke and Wohlfahrt that for congruence

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subgroups the two definitions coincide. In this paper we define the concept of level for arbitrary subgroups Γ of finite index in the special linear group $\mathrm{SL}_2(A_S)$, where A_S is the ring of S-integers of a global field k provided that k is an algebraic number field, or card $(S) \geq 2$. It is shown that this concept agrees with the usual notion of 'Stufe' for congruence subgroups. This provides an effective criterion for deciding whether or not an arbitrary subgroup of finite index is a congruence subgroup.

More precisely, let k be a global field, and let A_S be the ring of S-integers for a given finite non-empty set S of places of k containing the archimedean ones. For a non-zero ideal $\mathfrak{a}$ of A_S the principal S-congruence subgroup $\Gamma(\mathfrak{a})$ of level $\mathfrak{a}$ is defined as the kernel of the surjective homomorphism

$$\mathrm{res}_{\mathfrak{a}}\colon \mathrm{SL}_2(A_S) \longrightarrow \mathrm{SL}_2(A_S/\mathfrak{a})$$

obtained by taking residue classes mod $\mathfrak{a}$. A subgroup Δ of $\mathrm{SL}_2(A_S)$ of finite index is called an S-congruence subgroup if there exists a non-zero ideal $\mathfrak{q}$ in A_S such that $\Gamma(\mathfrak{q}) \subseteq \Delta$. An S-congruence subgroup Δ is defined to be of level $\mathfrak{q}$ ('Stufe') if $\Gamma(\mathfrak{q}) \subseteq \Delta$ and if $\mathfrak{q}$ is the largest ideal for which this inclusion is valid. The congruence subgroup problem for SL_2 is the question whether every subgroup of finite index in $\mathrm{SL}_2(A_S)$ is an S-congruence subgroup.

Serre [18] formulated the problem by considering the short exact sequence

$$1 \longrightarrow C^S(\mathrm{SL}_2) \longrightarrow \widehat{\mathrm{SL}_2(k)} \xrightarrow{\pi} \overline{\mathrm{SL}_2(k)} \longrightarrow 1$$

involving the respective completions $\widehat{\mathrm{SL}_2(k)}, \overline{\mathrm{SL}_2(k)}$ of $\mathrm{SL}_2(k)$ in the topology whose fundamental system of neighborhoods of 1 consists of all normal subgroups of finite index (resp. all congruence subgroups) in $\mathrm{SL}(A_S)$. The congruence kernel $C^S(\mathrm{SL}_2)$ is, by definition, the kernel of the homomorphism π, induced from the identity. Thus, the group $C^S(\mathrm{SL}_2)$ measures deviation from an affirmative answer to the congruence subgroup problem. It was proved by Serre [18], in the case card $(S) > 1$, that $C^S(\mathrm{SL}_2) \cong \mu_k$ (the finite group of roots of unity in k) if k is a totally imaginary number field and S is the set of all archimedean places, and $C^S(\mathrm{SL}_2) = 1$ otherwise. When card $(S) = 1, C^S(\mathrm{SL}_2)$ is infinite.

Probably, in the case of the group $\mathrm{SL}_2(\mathbb{Z})$, it was first stated by Klein in 1879 that there are subgroups of finite index in $\mathrm{SL}_2(\mathbb{Z})$ that are not congruence subgroups (see [8], p. 308, 418, 659); examples were given in 1887 by Fricke [2] and Pick [14].

Pursuing an idea of Fricke the concept of level for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ was extended to arbitrary subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index by

Wohlfahrt [21, 22]. The definition can be reformulated in group theoretical terms as follows: For a given natural number $n > 0$, let $Q(n)$ be the normal closure of the cyclic group $\langle u^n \rangle$ in $SL_2(\mathbb{Z})$,

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the level of Γ is defined to be the smallest positive integer n_Γ such that $Q(n_\Gamma) \leq \Gamma$. As a consequence of the relation

$$Q(q)\Gamma(n) = \Gamma(q),$$

where q is any divisor of n (see, e.g., [22], Theorem 2, or [12], Theorem VIII. 8), the two notions of level coincide for congruence subgroups.

In turn, this result, due to Fricke and Wohlfahrt, played a basic role in constructing and detecting families of non-congruence subgroups in $SL_2(\mathbb{Z})$ (see, e.g., [22], [15], [11], Theorem 4, [12], VIII. 18, [13]). The minimal index of a non–congruence subgroup in $SL_2(\mathbb{Z})$ is 7.

In section 2, we define the concept of level for arbitrary subgroups Γ of finite index in $SL_2(A_S)$ provided that k is an algebraic number field, or card $(S) \geq 2$. Let $\mathcal{J}(\Gamma)$ be the set of all non-zero ideals $\mathfrak{a}$ in A_S such that the normal closure $Q(\mathfrak{a})$ of the set of all unipotent matrices

$$M_\mathfrak{a} := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathfrak{a} \right\}$$

in $SL_2(A_S)$ is contained in Γ. Then the level of Γ is defined to be the minimal element $\mathfrak{a}_\Gamma$ in $\mathcal{J}(\Gamma)$. Next, for a given Γ of level $\mathfrak{a}_\Gamma$ it is proved that Γ is a congruence subgroup if and only if the principal congruence subgroup $\Gamma(\mathfrak{a}_\Gamma)$ is contained in Γ. The proof relies on the identity $\langle \Gamma(\mathfrak{b}),\ Q(\mathfrak{a}) \rangle = \Gamma(\mathfrak{a})$ if $\mathfrak{b} \subset \mathfrak{a}$, $\mathfrak{b} \neq 0$, $\mathfrak{a}, \mathfrak{b}$ ideals, which, in turn, is a consequence of a result of Bass (as was pointed out by J–P. Serre after we had worked out our proof with some congruence arguments).

A more general definition of the level of a subgroup (not necessarily of finite index) of $SL_n(D)$, D a Dedekind ring, is in [10], see Remark 2.6. (2).

In section 3, in the case $SL_2(\mathcal{O})$, $\mathcal{O}$ the ring of integers of an imaginary quadratic number field k, this criterion for deciding whether or not an arbitrary subgroup of finite index is a congruence subgroup is used to determine the minimum of the indices of non-congruence subgroups. We define, given $k = \mathbb{Q}(\sqrt{d})$, $d < 0$, d square free,

$$ncs(d) := \min \left\{ [SL_2(\mathcal{O}_d) : \Delta] \;\middle|\; \begin{array}{c} \Delta \text{ is a non-congruence subgroup of} \\ \text{finite index in } SL_2(\mathcal{O}_d) \end{array} \right\}.$$

Then we obtain the following values $ncs(-1) = 5$, $\mathrm{ncs}(-2) = 4$, $\mathrm{ncs}(-3) = 22$, $\mathrm{ncs}(-7) = 3$ and $\mathrm{ncs}(d) = 2$ in all other cases.

2. An extended concept of level

2.1. Let k be a global field, i.e. k is a finite extension of $\mathbb{Q}$, or the quotient field of a polynomial ring $\mathbb{F}_d[X], \mathbb{F}_d$ a finite field. The field k is an algebraic number field in the first case and a function field in the second. The set of places of k will be denoted by V, and V_∞ (resp. V_f) refers to the set of archimedean (resp. non-archimedean) places of k. If k is a function field, then V_∞ is empty. The completion of k at a place $v \in V$ is denoted by k_v. For a given place $v \in V$ the normalized absolute value on k_v is denoted by $||_v$. Let S be a finite non-empty set of places of k containing V_∞. Thus $S = S_\infty \cup S_f$, where $S_\infty = V_\infty$ and $S_f = S \cap V_f$. The set

$$A_S := \{x \in k \mid |x|_v \geq 0 \text{ for all } v \in V \setminus S\}$$

is a subring of k called the ring of S-integers of k. This ring is a Dedekind ring domain, and its field of quotients is k. Usually it is called the Dedekind ring of arithmetic type associated to S or a Hasse domain. If k is an algebraic number field and $S = S_\infty$, then the Hasse domain A_S is the ring of integers $\mathcal{O}$ of k.

Given a Dedekind ring A of arithmetic type one can consider subrings R which contain a non-zero ideal. The ring R is of finite index in A, and R contains a unique maximal A-ideal $\mathfrak{f} = \{r \in R \mid rA \subseteq R\}$ called the conductor of R. For any non-zero ideal $\mathfrak{r}$ of R the quotient ring $R/\mathfrak{r}$ is finite. One can show by use of some results in [17] that the class of finitely generated subrings of global fields coincides with the class of rings R introduced here.

2.2. Let $A = A_S$ be a Dedekind ring of arithmetic type associated to S, and let $\mathrm{SL}_2(A)$ be the special linear group of (2×2)-matrices of determinant one with entries in A. Given a non-zero ideal $\mathfrak{a}$ of A the principal S-congruence subgroup $\Gamma(\mathfrak{a})$ of level $\mathfrak{a}$ is defined as the kernel of the natural homomorphism $\mathrm{res}_\mathfrak{a}\colon \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathfrak{a})$ obtained by restriction mod $\mathfrak{a}$. This morphism is surjective ([1], Cor. 5.2.), thus $\Gamma(\mathfrak{a})$ is a normal subgroup of finite index in $\mathrm{SL}_2(A) =: \Gamma_A$.

Let $Q(\mathfrak{a})$ be the smallest normal subgroup of $\mathrm{SL}_2(A)$ containing the set of unipotent matrices

$$M_\mathfrak{a} := \left\{ u(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathfrak{a} \right\},$$

i.e., $Q(\mathfrak{a})$ is the normal closure of $M_{\mathfrak{a}}$ in $\mathrm{SL}_2(A)$. We observe the inclusion $Q(\mathfrak{a}) \subset \Gamma(\mathfrak{a})$.

A subgroup Δ of Γ_A of finite index is called a congruence subgroup if there exists a non-zero ideal $\mathfrak{a}$ in A such that $\Gamma(\mathfrak{a}) \subseteq \Delta$.

2.3 LEMMA: *Let Γ be an arbitrary subgroup of $\mathrm{SL}_2(A)$ of finite index, and let $\mathcal{J}(\Gamma)$ be the set of all non-zero ideals $\mathfrak{a}$ in A such that the group $Q(\mathfrak{a})$ is contained in Γ. Then:*

(1) *If the global field k is an algebraic number field, or* card $(S) \geq 2$, *then the set $\mathcal{J}(\Gamma)$ is non-empty.*
(2) *If $\mathfrak{a}, \mathfrak{b}$ are elements in $\mathcal{J}(\Gamma)$ then their greatest common divisior* $\gcd(\mathfrak{a}, \mathfrak{b})$ *is in $\mathcal{J}(\Gamma)$.*
(3) *If $\mathcal{J}(\Gamma)$ is non-empty, then there exists a uniquely determined minimal element $\mathfrak{a}_\Gamma$ in $\mathcal{J}(\Gamma)$. The minimal element $\mathfrak{a}_\Gamma$ in $\mathcal{J}(\Gamma)$ is called the level of Γ.*

Proof: (1) For this assertion we refer to 1.4. Proposition 1 in [18]. Note that the number field case is easy.

(2) Let $\mathfrak{c}$ be the greatest common divisor of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{J}(\Gamma)$. As abelian group $\mathfrak{c}$ is generated by $\mathfrak{a}$ and $\mathfrak{b}$. Given an ideal $\mathfrak{m}$ in A the group $Q(\mathfrak{m})$ is generated by the elements $\gamma u(x)\gamma^{-1}, x \in \mathfrak{m}, \gamma \in \Gamma_A$. Therefore, the inclusions $Q(\mathfrak{a}) \subseteq \Gamma$ and $Q(\mathfrak{b}) \subseteq \Gamma$ imply that $Q(\mathfrak{c}) \subseteq \Gamma$.

(3) The existence of a minimal element in $\mathcal{J}(\Gamma)$ with respect to the natural ordering is immediate.

2.4 Remark: In the case where k is a function field and card $(S) = 1$ one can show by looking at [18] section 3 that there exist uncountably many subgroups of finite index in $\mathrm{SL}_2(A)$ not containing any $Q(\mathfrak{a})$.

2.5 THEOREM: *Let Γ be a subgroup of $\mathrm{SL}_2(A)$ of finite index, let $\mathfrak{a}_\Gamma$ be the level of Γ. Then the following assertions are equivalent:*

(1) *There exists a non-zero ideal $\mathfrak{a}$ in A such that $\Gamma(\mathfrak{a}) \subseteq \Gamma$, i.e. Γ is a congruence subgroup.*
(2) *The principal congruence subgroup $\Gamma(\mathfrak{a}_\Gamma)$ is contained in Γ.*

Proof: We prove that assertion (1) implies (2); the reverse implication being obvious. The group Γ has level $\mathfrak{a}_\Gamma$, i.e., by definition, $Q(\mathfrak{a}_\Gamma) \subseteq \Gamma$. On the other hand, by $\Gamma(\mathfrak{a}) \subset \Gamma$, one has $\Gamma(\mathfrak{a} \cdot \mathfrak{a}_\Gamma) \subset \Gamma$. We show that the subgroup generated by $\Gamma(\mathfrak{a} \cdot \mathfrak{a}_\Gamma)$ and $Q(\mathfrak{a}_\Gamma)$ is equal to $\Gamma(\mathfrak{a}_\Gamma)$, i.e.

$$\langle \Gamma(\mathfrak{a} \cdot \mathfrak{a}_\Gamma), Q(\mathfrak{a}_\Gamma) \rangle = \Gamma(\mathfrak{a}_\Gamma). \tag{3}$$

This implies $\Gamma(\mathfrak{a}_\Gamma) \subset \Gamma$. This equality is a consequence of a result of Bass (as we learned from J-P. Serre after we had completed our original proof with some detailed congruence arguments). Indeed, we have to prove that $\langle \Gamma(\mathfrak{b}), Q(\mathfrak{a})\rangle = \Gamma(\mathfrak{a})$ if $\mathfrak{b} \subset \mathfrak{a}$, $\mathfrak{b} \neq 0$, $\mathfrak{a}, \mathfrak{b}$ ideals. Passing over to the semi-local ring $A/\mathfrak{b}$ this amounts to showing that the congruence subgroup modulo $\mathfrak{a}$ of SL_2 is equal to $Q(\mathfrak{a})$. But this is a special case of Proposition 5.1. in [1].

For the readers convenience we include our original elementary argument.

First, $\langle \Gamma(\mathfrak{a} \cdot \mathfrak{a}_\Gamma),\, Q(\mathfrak{a}_\Gamma)\rangle \subseteq \Gamma(\mathfrak{a}_\Gamma)$. To see the other inclusion, we are going to multiply a given element γ in $\Gamma(\mathfrak{a}_\Gamma)$ with elements in $Q(\mathfrak{a}_\Gamma)$ to bring it to $\Gamma(\mathfrak{a}\cdot\mathfrak{a}_\Gamma)$. Let a be any element in $\mathfrak{a}_\Gamma$; then the following matrices are defined:

$$(4) \qquad u(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad v(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},$$

$$(5) \qquad w(x,y,a) = \begin{pmatrix} 1+xya & -x^2a \\ y^2a & 1-xya \end{pmatrix}, \quad x, y \in A, \quad (x,y) = 1.$$

Let $r, s \in A$ be elements with $xr - ys = 1$; then we see

$$w(x,y,a) = \begin{pmatrix} x & s \\ y & r \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & s \\ y & r \end{pmatrix}^{-1}.$$

A given element γ in $\Gamma(\mathfrak{a}_\Gamma)$ is of the form

$$(6) \qquad \gamma = \begin{pmatrix} 1+a_1 & a_2 \\ a_3 & 1+a_4 \end{pmatrix} \quad \text{with } a_i \in \mathfrak{a}_\Gamma, \quad 1 \leq i \leq 4.$$

First, there is an $a \in \mathfrak{a}_\Gamma$ such that the ideals $(1 + a_1 + aa_3)$ and $\mathfrak{a}$ do not have a common divisor. To see this we solve the congruences

$$(7) \qquad \begin{aligned} &a \equiv 0 \bmod \mathfrak{a}_\Gamma, && \\ &a \equiv 0 \bmod \mathfrak{p} && \text{if } \mathfrak{p}|\mathfrak{a} \quad \text{and } \mathfrak{p}|a_3, \\ &a \equiv -a_3^{-1}a_1 \bmod \mathfrak{p} && \text{if } \mathfrak{p}|\mathfrak{a} \quad \text{and } \mathfrak{p} \nmid a_3. \end{aligned}$$

Since $\mathfrak{a}_\Gamma$ divides a_3 there is a solution a to these congruences. Then one checks that all prime ideals $\mathfrak{p}$ dividing $\mathfrak{a}$ do not divide the ideal $(1 + a_1 + aa_3)$. By multiplication with $u(a)$ we obtain from $\gamma \in \Gamma(\mathfrak{a}_\Gamma)$ the element

$$(8) \qquad \gamma_1 = u(a) \cdot \gamma = \begin{pmatrix} 1+b_1 & b_2 \\ b_3 & 1+b_4 \end{pmatrix} \in \Gamma(\mathfrak{a}_\Gamma)$$

with $b_i \in \mathfrak{a}_\Gamma$ and such that the ideals $(1 + b_1)$ and $\mathfrak{a}$ do not have a common divisor.

Second, consider the matrix $w(x,y,z) \cdot \gamma_1$. Its $(1,1)$-entry is given as

$$(1+b_1)(1+xyz) - x^2 z b_2. \tag{9}$$

We are going to determine the variables x, y, z in such a way that the congruence

$$(1+b_1)(1+xyz) - x^2 z b_2 \equiv 1 \bmod \mathfrak{a}\mathfrak{a}_\Gamma \tag{10}$$

holds. Indeed, this congruence is equivalent to

$$b_1 - b_2 x^2 z + (1+b_1)\, xyz \equiv 0 \bmod \mathfrak{a} \cdot \mathfrak{a}_\Gamma. \tag{11}$$

By choosing $z = b_1$ and $x = 1$, this leads to the congruence

$$b_1(1 - b_2 + (1+b_1)y) \equiv 0 \bmod \mathfrak{a} \cdot \mathfrak{a}_\Gamma. \tag{12}$$

Recall that $b_1 \in a_\Gamma$; thus we can solve (12) if we can find a solution for

$$(1 - b_2 + (1+b_1)y) \equiv 0 \bmod \mathfrak{a}. \tag{13}$$

In turn, this congruence can be solved in y because $(1+b_1)$ and $\mathfrak{a}$ do not have a common divisor by the first step (see (8)).

Third, choosing x, y, z in such a way that (10) holds, we obtain the matrix

$$\gamma_2 = w(x,y,z) \cdot \gamma_1 = \begin{pmatrix} 1+c_1 & c_2 \\ c_3 & 1+c_4 \end{pmatrix}$$

with $c_1 \in \mathfrak{a} \cdot \mathfrak{a}_\Gamma, c_j \in \mathfrak{a}_\Gamma, j = 2,3,4$. By multiplying γ_2 with suitable elements of the form $u(\alpha)$ and $v(\beta)$ on the left and the right hand side we can achieve that c_2 and c_3 are elements in $\mathfrak{a} \cdot \mathfrak{a}_\Gamma$. The determinant condition $\det(\gamma_2) = 1$ gives $c_4 \in \mathfrak{a} \cdot \mathfrak{a}_\Gamma$. This proves assertion (3).

2.6 Remarks: (1) The congruence kernel $C^S(\mathrm{SL}_2)$ measures deviation from a positive answer to the congruence subgroup problem. By [18], $C^S(\mathrm{SL}_2)$ is infinite if and only if card $(S) = 1$.

There exist three families of Dedekind rings of arithmetic type with card $(S) = 1$.

(I) Let k be the field of rational numbers, and $A = \mathbb{Z}$ is the ring of integers in $\mathbb{Q}$.

(II) Let $k = \mathbb{Q}(\sqrt{d}), d < 0, d$ square free, an imaginary quadratic number field, and $A = \mathcal{O}_d$ the ring of algebraic integers in k.

(III) Let k be a function field given as the quotient field of the coordinate ring A of an affine curve obtained by removing a point from a projective curve defined over a finite field.

The result 2.5 provides an effective criterion for deciding whether or not an arbitrary subgroup of finite index in the cases (I) and (II) is a congruence subgroup. It can be used to detect classes of examples of non-congruence subgroups, constructed, e.g., in case (II) by use of [3]. In case (II) our criterion will also be used in the next section.

(2) The referee pointed out that a more general definition of the level of a subgroup (not necessarily of finite index) of $\mathrm{SL}_n(D)$ where D is a Dedekind ring has been given in [10], p. 257. A more general version of our result 2.5 (together with equation (3)) appears as Corollaries 1.2 and 1.3 in this paper.

This definition of level is more general than that given here in another respect. It only requires that a subgroup contains the normal subgroup of $E_n(D)$ generated by the $\mathfrak{q}$-elementary matrices, where $\mathfrak{q}$ is a D-ideal. For congruence subgroups these definitions coincide. This is due to the fact that a result such as 2.5 holds if one replaces $Q(\mathfrak{a})$ by the normal subgroup of $E_2(A)$ generated by $M_{\mathfrak{a}}$.

3. Non-congruence subgroups of minimal index in the imaginary quadratic number field case

Let d be a squarefree negative integer and $\mathcal{O}_d$ the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$. The group $\mathrm{SL}_2(\mathcal{O}_d)$ contains non-congruence subgroups of finite index (cf. 2.6). We define

$$(14)\quad \mathrm{ncs}(d) := \min\left\{ [\mathrm{SL}_2(\mathcal{O}_d) : \Delta] \;\middle|\; \begin{array}{c} \Delta \text{ is a non-congruence subgroup} \\ \text{of finite index in } \mathrm{SL}_2(\mathcal{O}_d) \end{array} \right\}.$$

The exact values of ncs are described in the following:

3.1 PROPOSITION: *We have*

$$\mathrm{ncs}(-1) = 5, \quad \mathrm{ncs}(-2) = 4, \quad \mathrm{ncs}(-3) = 22, \quad \mathrm{ncs}(-7) = 3$$

and $\mathrm{ncs}(d) = 2$ *in all other cases.*

Proof:* For a given d as above let $1, \omega$ be a $\mathbb{Z}$-basis of $\mathcal{O}_d$ and put

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_d).$$

* In the proof certain computer calculations are needed. These were done using the computer algebra program GAP developed by J. Neubüser, Lehrstuhl D at the RWTH Aachen.

The subgroup generated by the images of T, U in the commutator quotients $\mathrm{SL}_2(\mathcal{O}_d)^{ab}$ has $\mathbb{Z}$-rank 0 or 1. This is implied by Théorème 9 of [18]. Now we suppose

$$\mathrm{rk}_{\mathbb{Z}}(\mathrm{SL}_2(\mathcal{O}_d)^{ab}) \geq 2. \tag{15}$$

Then there is a surjective homomorphism φ: $\mathrm{SL}_2(\mathcal{O}_d) \to \mathbb{Z}/2\mathbb{Z}$ with $\varphi(T) = \varphi(U) = 0$. Let Δ be the kernel of φ. By construction, the level of Δ is the unit ideal $\mathcal{O}_d$. We infer from Theorem 2.5 that Δ is not a congruence subgroup.

Hypothesis (15) is verified if the class number of $\mathcal{O}_d$ is greater than or equal to 2. This follows from Corollaire 3 of Théorème 8 in [18]. Hence Proposition 3.1. is proved for all d not in the following list,

$$-1, -2, -3, -7, -11, -19, -43, -67, -163.$$

Of course these are exactly those values of d for which $\mathcal{O}_d$ has class number 1.

For $d = -43, -67$ hypothesis (15) is verified in [20]. For an explicit statement of hypothesis (15) for the case $d = -163$, see p. 631 in [16]. In the remaining six cases a more direct method is needed.

THE CASE $d = -19$: A presentation of $\mathrm{SL}_2(\mathcal{O}_d)$ is contained in [19]. We find $\mathrm{SL}_2(\mathcal{O}_d)^{ab} = \mathbb{Z}$. Hence there is a surjective homomorphism φ: $\mathrm{SL}_2(\mathcal{O}_d)^{ab} \to \mathbb{Z}/2\mathbb{Z}$. Let Δ be the kernel of φ. From [19] we find $U \notin \Delta$, hence the level of Δ is $2 \cdot \mathcal{O}_{-19}$. Note that the prime 2 is inertial in $\mathbb{Q}(\sqrt{-19})$. If $\Gamma(2 \cdot \mathcal{O}_{-19})$ would be contained in Δ then $\mathrm{SL}_2(\mathcal{O}_{-19}/2 \cdot \mathcal{O}_{-19})$ would have a subgroup of index 2. But $\mathcal{O}_{-19}/2 \cdot \mathcal{O}_{-19}$ is the field with 4 elements, and Satz 6.10 of [6] says that this is not the case. By Theorem 2.5 we infer that Δ is not a congruence subgroup.

THE CASE $d = -11$: In this case exactly the same argument as for $d = -19$ may be applied to get $\mathrm{ncs}(-11) = 2$.

THE CASE $d = -7$: From the presentation given in [19] or [5] we read off that the group $\mathrm{SL}_2(\mathcal{O}_{-7})$ has exactly 3 subgroups of index 2. The prime 2 is decomposed in $\mathbb{Q}(\sqrt{-7})$; let $2\mathcal{O}_{-7} = \mathfrak{p} \cdot \bar{\mathfrak{p}}$ be the corresponding primary decomposition. We have

$$\mathrm{SL}_2(\mathcal{O}_{-7}/2\mathcal{O}_{-7}) = \mathrm{SL}_2(\mathcal{O}_{-7}/\mathfrak{p}) \times \mathrm{SL}_2(\mathcal{O}_{-7}/\bar{\mathfrak{p}}) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}).$$

The group on the right hand side evidently has 3 subgroups of index 2. Since the reduction homomorphism $\mathrm{SL}_2(\mathcal{O}_{-7}) \to \mathrm{SL}_2(\mathcal{O}_{-7}/2\mathcal{O}_{-7})$ is surjective the 3 subgroups of index 2 in $\mathrm{SL}_2(\mathcal{O}_{-7})$ all contain $\Gamma(2 \cdot \mathcal{O}_{-7})$. We infer $\mathrm{ncs}(-7) \geq 3$. Is is also clear (from the presentation in [19]) that $\mathrm{SL}_2(\mathcal{O}_{-7})$ has a normal subgroup

Δ of index 3. The level of Δ has to be $3 \cdot \mathcal{O}_{-7}$ because $3 \cdot \mathcal{O}_{-7}$ is a prime ideal. If $\Gamma(3 \cdot \mathcal{O}_{-7})$ is contained in Δ then $\mathrm{SL}_2(\mathcal{O}_{-7}/3\mathcal{O}_{-7}) = \mathrm{SL}_2(\mathbb{F}_9)$ ($\mathbb{F}_9$ the field with 9 elements) contains a normal subgroup of index 3. By Satz 6.10 of [6] this is not the case. We conclude that Δ is not a congruence subgroup and $\mathrm{ncs}(-7) = 3$.

THE CASE $d = -2$: Using the low-index-subgroup routine provided by GAP on the presentation for $\mathrm{SL}_2(\mathcal{O}_d)$ from [19] or [5] we find the following table for the numbers of conjugacy classes $N(I)$ of subgroups of index $I \mathrm{in} \mathrm{SL}_2(\mathcal{O}_d)$:

I	$N(I)$	$l(I)$
1	1	1
2	3	2
3	5	$3\ \omega$
4	9	12

The entry $l(I)$ gives the maximal ideal $\mathfrak{a} = l(I) \cdot \mathcal{O}_{-2}$ so that $Q(\mathfrak{a}) \leq \Delta$ for every subgroup Δ of index I in $\mathrm{SL}_2(\mathcal{O}_{-2})$. Here we have chosen $\omega = \sqrt{-2}$. For every subgroup Δ of index I in $\mathrm{SL}_2(\mathcal{O}_{-2})$ the computer program GAP provides us with the permutation representation φ_Δ: $\mathrm{SL}_2(\mathcal{O}_{-2}) \longrightarrow \mathcal{S}_I$ ($\mathcal{S}_I$ the symmetric group on I elements) on the set of right cosets $\mathrm{SL}_2(\mathcal{O}_{-2})/\Delta$. From this information the level of Δ can easily be read off. For one of the subgroups Δ of index 4 GAP gives

$$\begin{aligned}
\varphi_\Delta : T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &\longmapsto\ & (2,3,4), \\
U &= \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} &\longmapsto\ & (2,3,4), \\
A &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &\longmapsto\ & (1,2)(3,4), \\
J &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &\longmapsto\ & \mathrm{id}\,.
\end{aligned}$$

The subgroup Δ is the stabilizer of 1 under this representation. We find that $T^3, U^3, TU^{-1} \in \ker \varphi_\Delta$, hence the level of Δ is $\mathfrak{a} = (1-\omega)\cdot\mathcal{O}_{-2}$. As an application of the Todd–Coxeter routine of GAP we find that the group

$$\mathrm{SL}_2(\mathcal{O}_{-2})/ \ll T^3, U^3, TU^{-1} \gg$$

has 24 elements. Here the double brackets $\ll\ \gg$ stand for the normal subgroup generated by the elements included. This implies that

$$\ll T^3, U^3, TU^{-1} \gg = \Gamma_{-2}(\mathfrak{a})$$

where $\Gamma_{-2}(\mathfrak{a})$ is the full congruence group of level $\mathfrak{a} = (1-\omega)\cdot\mathcal{O}_{-2}$ in $\mathrm{SL}_2(\mathcal{O}_{-2})$. We conclude that Δ, and hence every group in the conjugacy class of Δ, is a congruence subgroup. Similarly there is a conjugacy class of subgroups of $\mathrm{SL}_2(\mathcal{O}_{-2})$ of index 4 and level $(1+\omega)\mathcal{O}_{-2}$ arising from the isomorphism

$$\mathrm{SL}_2(\mathcal{O}_{-2}/(1+\omega)\mathcal{O}_{-2}) \cong \mathrm{SL}_2(\mathbb{F}_3).$$

GAP tells us that there is a third conjugacy class of subgroups of $\mathrm{SL}_2(\mathcal{O}_{-2})$ with index 4 and level dividing 3. We give the permutation representation of one of its repesentatives:

$$\begin{aligned} \varphi_\Delta : T &\longmapsto (2,3,4),\\ U &\longmapsto \mathrm{id},\\ A &\longmapsto (1,2)(3,4),\\ J &\longmapsto \mathrm{id}. \end{aligned}$$

There are two ways to see that Δ is not a congruence subgroup. Since Δ has obviously level $3\cdot\mathcal{O}_{-2}$, our Theorem 2.5 implies that $\Gamma_{-2}(3\cdot\mathcal{O}_{-2}) \leq \Delta$. Considering also the 2 conjugacy classes described before we find that

$$\mathrm{SL}_2(\mathcal{O}_{-2}/3\cdot\mathcal{O}_{-2}) \cong \mathrm{SL}_2(\mathbb{F}_3)\times\mathrm{SL}_2(\mathbb{F}_3)$$

would have at least 3 conjugacy classes of subgroups of index 4. It is a simple exercise to see that this group has only two such conjugacy classes. The second method first notices:

$$w = TUATU^{-1}A^{-1}U^{-1}T^{-1}AUT^{-1}A^{-1} \in \Gamma_{-2}(3\cdot\mathcal{O}_{-2}).$$

By Theorem 2.5 we conclude $\varphi_\Delta(w) = \mathrm{id}$. A quick glance at the above permutation representation shows that this is not the case.

GAP tells us that in addition to the 3-conjugacy classes described so far there are 6 conjugacy classes of subgroups of $\mathrm{SL}_2(\mathcal{O}_{-2})$ of index 4 and level dividing $4\cdot\mathcal{O}_{-2}$. But the finite group $\mathrm{SL}_2(\mathcal{O}_{-2}/4\cdot\mathcal{O}_{-2})$ has only 4 conjugacy classes of subgroups of index 4. Hence there are 2 conjugacy classes of non-congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_{-2})$ of index 4 and level dividing $4\cdot\mathcal{O}_{-2}$. These are represented by the two distinct normal subgroups of index 4 and cyclic quotient. That these are non-congruence subgroups is shown by Theorem 2.5 and

$$\mathrm{SL}_2(\mathcal{O}_{-2}/4\cdot\mathcal{O}_{-2})^{ab} = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

The discussion so far shows that $\mathrm{ncs}(-2) \leq 4$. It remains to treat the subgroups of index 2, 3. A simple computation (made easier by the help of GAP) shows that

$$\mathrm{SL}_2(\mathcal{O}/2\cdot\mathcal{O}_{-2})^{ab} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

This immediately implies that all subgroups of index $I = 2$ in $\mathrm{SL}_2(\mathcal{O}_{-2})$ are congruence groups.

It is elementary to show that

$$\mathrm{SL}_2(\mathcal{O}_{-2}/\omega\cdot\mathcal{O}_{-2}) \cong \mathrm{SL}_2(\mathbb{F}_2)$$

has 1 conjugacy class of subgroups of index 3 and

$$\mathrm{SL}_2(\mathcal{O}_{-2}/2\mathcal{O}_{-2}) \cong \mathrm{SL}_2(\mathbb{F}_3) \times \mathrm{SL}_2(\mathbb{F}_3)$$

has 4 of these. We conclude that $\mathrm{SL}_2(\mathcal{O}_{-2})$ has at least 5 conjugacy classes of congruence subgroups of index 3. Since GAP tells us that there are only 5 conjugacy classes of subgroups of index 3 in $\mathrm{SL}_2(\mathcal{O}_{-2})$ all these are congruence subgroups.

THE CASE $d = -1$: With the same notation as for the case $d = -2$ we give the table of conjugacy classes of subgroups of index ≤ 5 in $\mathrm{SL}_2(\mathcal{O}_{-1})$ computed by GAP:

I	$N(I)$	$l(I)$
1	1	1
2	3	2
3	1	1 + i
4	4	4
5	4	5.6

The situation is now analysed similarly to the case $d = -2$. First of all we consider the finite group $\mathrm{SL}_2(\mathcal{O}_{-1}/4\cdot\mathcal{O}_{-1})$. We find

$$\mathrm{SL}_2(\mathcal{O}_{-1}/4\mathcal{O}_{-1})^{ab} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

This shows that all subgroups of index 2 in $\mathrm{SL}_2(\mathcal{O}_{-1})$ are congruence subgroups. It is also easily shown that $\mathrm{SL}_2(\mathcal{O}_{-1}/4\mathcal{O}_{-1})$ has at least 1 conjugacy class of subgroups of index 3 and 4 of index 4. This then shows that all subgroups of indices 3, 4 in $\mathrm{SL}_2(\mathcal{O}_{-1})$ are congruence subgroups. There are 2 distinct conjugacy classes of congruence subgroups of index 5 in $\mathrm{SL}_2(\mathcal{O}_{-1})$ coming from the subgroups of index 5 in

$$\mathrm{SL}_2(\mathbb{F}_5) \cong \mathrm{SL}_2(\mathcal{O}_{-1}/(1+2i)\mathcal{O}_{-1}) \cong \mathrm{SL}_2(\mathcal{O}_{-1}/(1-2i)\mathcal{O}_{-1}).$$

There are 2 further conjugacy classes of subgroup of index 5 in $SL_2(\mathcal{O}_{-1})$. We give the permutation representations of a representative of each:

$$\begin{array}{rclcl} \varphi_{\Delta_1}: T & = & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \longmapsto & (3,4,5), \\ U & = & \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} & \longmapsto & (1,2), \\ A & = & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \longmapsto & (2,3)(4,5), \\ L & = & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} & \longmapsto & (4,5), \end{array} \qquad \begin{array}{rcl} \varphi_{\Delta_2}: T & \longmapsto & (4,5), \\ U & \longmapsto & (1,2,3), \\ A & \longmapsto & (3,4), \\ L & \longmapsto & (1,2). \end{array}$$

Again the subgroup Δ_1, Δ_2 are the stabilizers of 1 under the two representations respectively. The group Δ_1 has level $2 \cdot \mathcal{O}_{-1}$ whereas Δ_2 has level $3 \cdot \mathcal{O}_{-1}$. Neither can be a congruence subgroup since

$$\begin{aligned} SL_2(\mathcal{O}_{-1}/6\mathcal{O}_{-1}) &\cong SL_2(\mathcal{O}_{-1}/2\mathcal{O}_{-1}) \times SL_2(\mathcal{O}_{-1}\backslash 3\mathcal{O}_{-1}) \\ &\cong SL_2(\mathcal{O}_{-1}/2 \cdot \mathcal{O}_{-1}) \times SL_2(\mathbb{F}_9) \end{aligned}$$

has no subgroup of index 5.

THE CASE $d = -3$: With the same notation as for the case $d = -2$ we give the table of conjugacy classes of subgroups of index ≤ 22 in $SL_2(\mathcal{O}_{-3})$ computed by GAP:

I	$N(I)$	$l(I)$	I	$N(I)$	$l(I)$	I	$N(I)$	$l(I)$
1	1	1	9	1	3	17	0	
2	0		10	1	2	18	5	6
3	1	$1+2\omega$	11	0		19	0	
4	1	$1+2\omega$	12	7	12	20	10	$4.19 \cdot (1+2\omega)$
5	1	2	13	0		21	6	$7+14\omega$
6	2	$2+4\omega$	14	6	91	22	2	$5 \cdot 7$
7	4	7	15	4	6			
8	3	$7+14\omega$	16	6	28			

Here we have chosen $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. It is a series of simple exercises to analyse the subgroups of the finite groups $SL_2(\mathcal{O}_{-3}/l\mathcal{O}_{-3})$ where l runs through the divisors of the $l(I)$ given above for $1 \leq I \leq 21$ and find that there are enough conjugacy classes of congruence subgroups in $SL_2(\mathcal{O}_{-3})$ to exhaust all conjugacy classes of these indices. The computer program gives us 2 conjugacy classes of subgroups of index 22 in $SL_2(\mathcal{O}_{-1})$. We give the permutation representations of two representatives:

$$T=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad U=\begin{pmatrix}1&\omega\\0&1\end{pmatrix},\quad A=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad L=\begin{pmatrix}\omega^2&0\\0&\omega\end{pmatrix};$$

$$\begin{aligned}
\varphi_{\Delta_1}\colon T &\longmapsto (1,7,13,10,18,5,14)(2,9,19,16,12)(3,4,15,22,20)(6,21,8,11,17),\\
U &\longmapsto (1,18,7,5,13,14,20)(2,9,19,16,12)(3,22,4,20,15)(6,8,17,21,11),\\
A &\longmapsto (1,3)(2,5)(4,9)(7,11)(8,10)(12,16)(14,19)(15,18)(17,20)(21,22),\\
L &\longmapsto (1,5,7)(2,3,11)(4,6,9)(8,19,15)(10,18,14)(12,20,21)(16,22,17).\\
\varphi_{\Delta_2}\colon T &\longmapsto (1,7,14,5,15,11,19)(2,18,22,17,13)(3,20,4,21,9)(6,8,12,16,10),\\
U &\longmapsto (1,14,15,19,7,5,11)(2,17,18,13,22),(3,4,9,20,21)(6,8,12,16,10),\\
A &\longmapsto (1,3)(2,5)(4,10)(6,15)(7,12)(8,13)(9,16)(11,20)(14,17)(18,22),\\
L &\longmapsto (1,5,7)(2,3,12)(4,16,18)(6,17,20)(8,13,21)(9,10,22)(11,14,15).
\end{aligned}$$

The group Δ_1 (which is as always the stabilizer of 1 under the representation φ_{Δ_1}) has level $\mathfrak{a}_1 = 5\cdot(2+3\omega)\cdot\mathcal{O}_{-3}$ and Δ_2 has level $\mathfrak{a}_2 = 5(1+3\omega)\mathcal{O}_{-3}$. Neither can be a congruence subgroup since

$$\mathrm{SL}_2(\mathcal{O}_{-3}/\mathfrak{a}_1) \cong \mathrm{SL}_2(\mathcal{O}_{-3}/\mathfrak{a}_2) \cong \mathrm{SL}_2(\mathbb{F}_{25}) \times \mathrm{SL}_2(\mathbb{F}_7)$$

has no subgroup of index 22.

For the groups Δ_1, Δ_2 we have

$$\varphi_{\Delta_1}(\mathrm{SL}_2(\mathcal{O}_{-3})) = \varphi_{\Delta_2}(\mathrm{SL}_2(\mathcal{O}_{-3})) = A_{22}$$

which also implies that they are not congruence subgroups.

3.2 Remark: There is a subgroup Δ of index 12 in $\mathrm{SL}_2(\mathcal{O}_{-3})$ so that Δ acts fixed point freely on 3-dimensional hyperbolic space

$$H^3 = \mathrm{SL}_3(\mathbb{C})/\,\mathrm{SU}(2)$$

and so that H^3/Γ is homeomorphic to the complement of the figure-eight knot in the 3-sphere S^3. This was discovered by R. Riley; see [5] for a description of this group. Our discussion in the proof of Proposition 3.1 shows that Δ is a congruence subgroup (of level $4\cdot\mathcal{O}_{-3}$). This gives rise to an interesting algebra of endomorphisms of $\Delta^{ab} = \mathbb{Z}$ given by the usual Hecke-algebra construction. The eigenvalues of these Hecke operators are in intimate relationship with the traces of the Frobenius maps for the elliptic curve $y^2 = x^3 + (\omega-1)x^2 - \omega x$.

3.3 Remark: It is natural the consider also congruence subgroups in the groups

$$\mathrm{PSL}_2(\mathcal{O}_d) := \mathrm{SL}_2(\mathcal{O}_d)/\{\pm 1\}.$$

They are defined as the

$$P\Gamma(\mathfrak{a}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_d) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{a} \right\}$$

as $\mathfrak{a}$ varies over the nonzero ideals of $\mathcal{O}_d$. Let $\mathrm{npcs}(d)$ be the minimal index of a non-congruence subgroup in $\mathrm{PSL}_2(\mathcal{O}_d)$. We clearly have $\mathrm{npcs}(d) \geq \mathrm{ncs}(d)$ for all d. Small refinements in the arguments for Proposition 3.1 show that $\mathrm{npcs}(d) = \mathrm{ncs}(d)$ for all d.

3.4 Remark: Let $\Delta \leq \mathrm{SL}_2(\mathcal{O}_d)$ be a congruence subgroup and $\mathrm{ncs}(\Delta)$ the minimal index of a non-congruence subgroup in Δ. Let $\mathrm{ns}(d)$ be the minimum of the $\mathrm{ncs}(\Delta)$ as Δ varies over all congruence subgroups of $\mathrm{SL}_2(\mathcal{O}_d)$. By an application of the methods of the present paper and of the results of [3] it is possible to prove $\mathrm{ns}(d) = 2$ for all d. The same result holds for the groups $\mathrm{PSL}_2(\mathcal{O}_d)$.

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